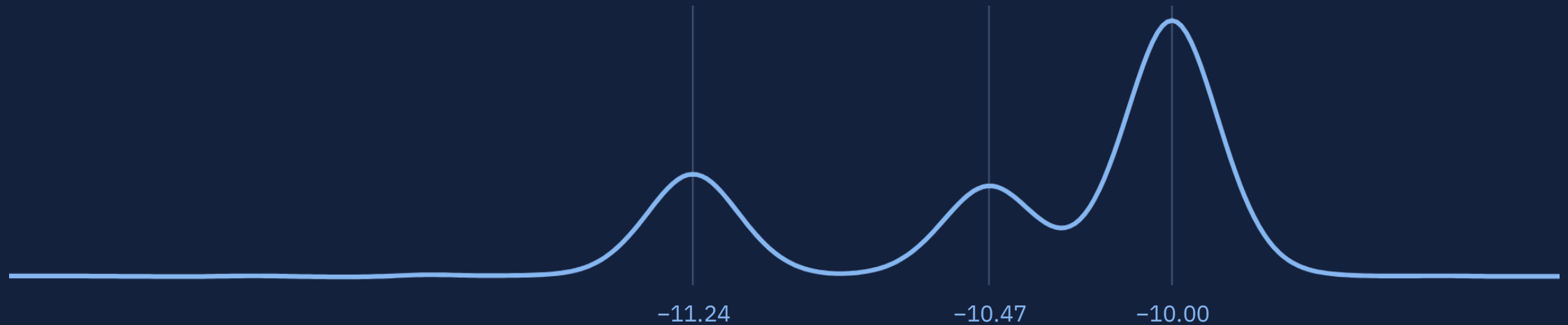


The Rodeo Algorithm

Finding energy levels with one ancilla qubit: implementation, cost and limits



THE PROBLEM

Energy levels anywhere in the spectrum

Classical

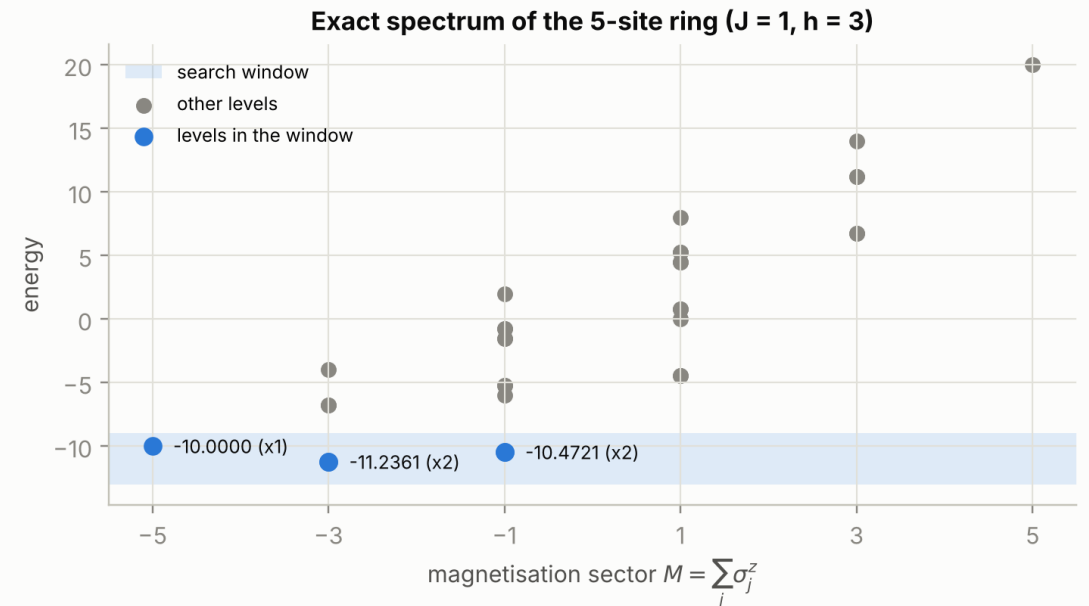
Exact diagonalisation: memory grows as 4^n . Lanczos: only the extreme levels. Interior levels: shift-invert, the slowest route.

Quantum

Phase estimation: deep controlled-U ladders plus a QFT. VQE: built for the ground state, with a hard classical optimisation.

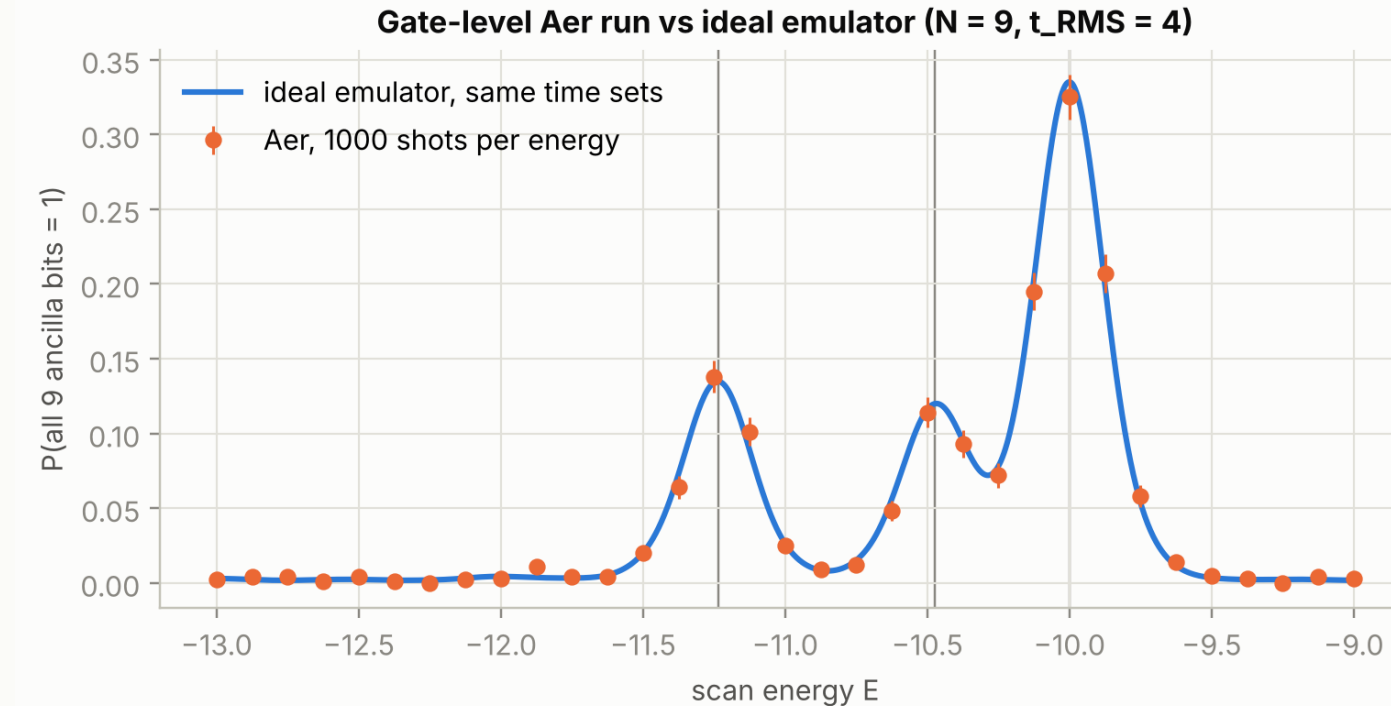
5-site Heisenberg ring in a field, $J = 1$, $h = 3$. Between $E = -13$ and -9 sit three levels: two doublets and one singlet.

$$H = J \sum_{\langle jk \rangle} \vec{\sigma}_j \cdot \vec{\sigma}_k + h \sum_j \sigma_j^z$$



HEADLINE RESULT

All three levels found on gate-level circuits



$$-11.226 \pm 0.007$$

$$\text{exact } -11.236 \cdot p = 0.136 \text{ (0.133)}$$

$$-10.464 \pm 0.010$$

$$\text{exact } -10.472 \cdot p = 0.112 \text{ (0.116)}$$

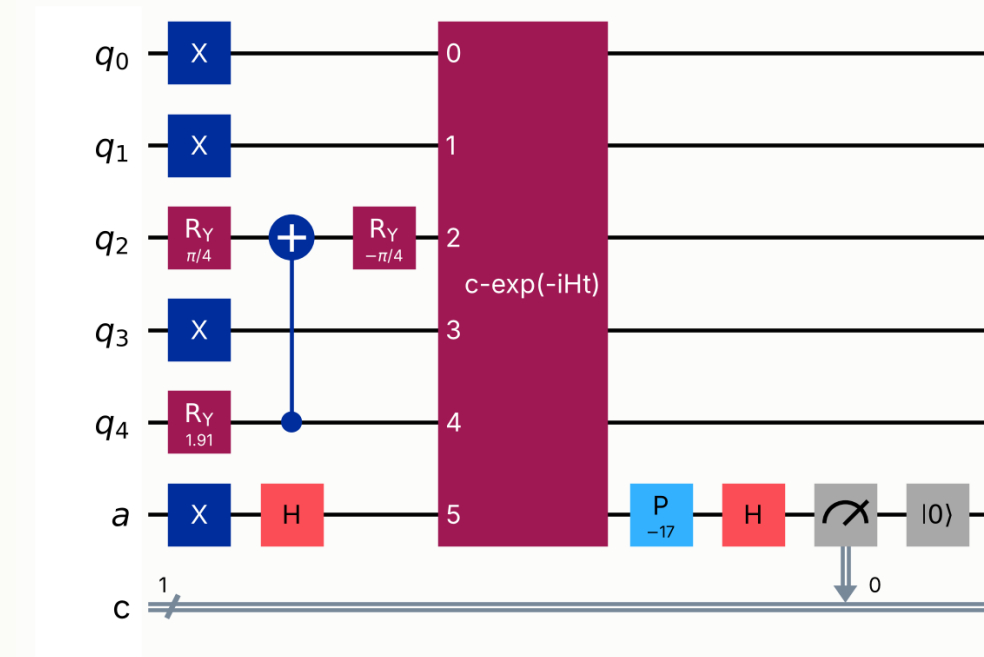
$$-9.992 \pm 0.005$$

$$\text{exact } -10.000 \cdot p = 0.329 \text{ (0.333)}$$

6 qubits, 9 mid-circuit measurements, 33 000 shots. Agrees with the ideal emulator: $\chi^2/\text{dof} = 1.19$.

HOW IT WORKS

An interferometer that lets one energy through



$$P_N(E) = \sum_j p_j \prod_{n=1}^N \cos^2 \left[\frac{(E_j - E) t_n}{2} \right]$$

1 • Kick-back. The controlled time evolution writes each level's energy into the ancilla's phase.

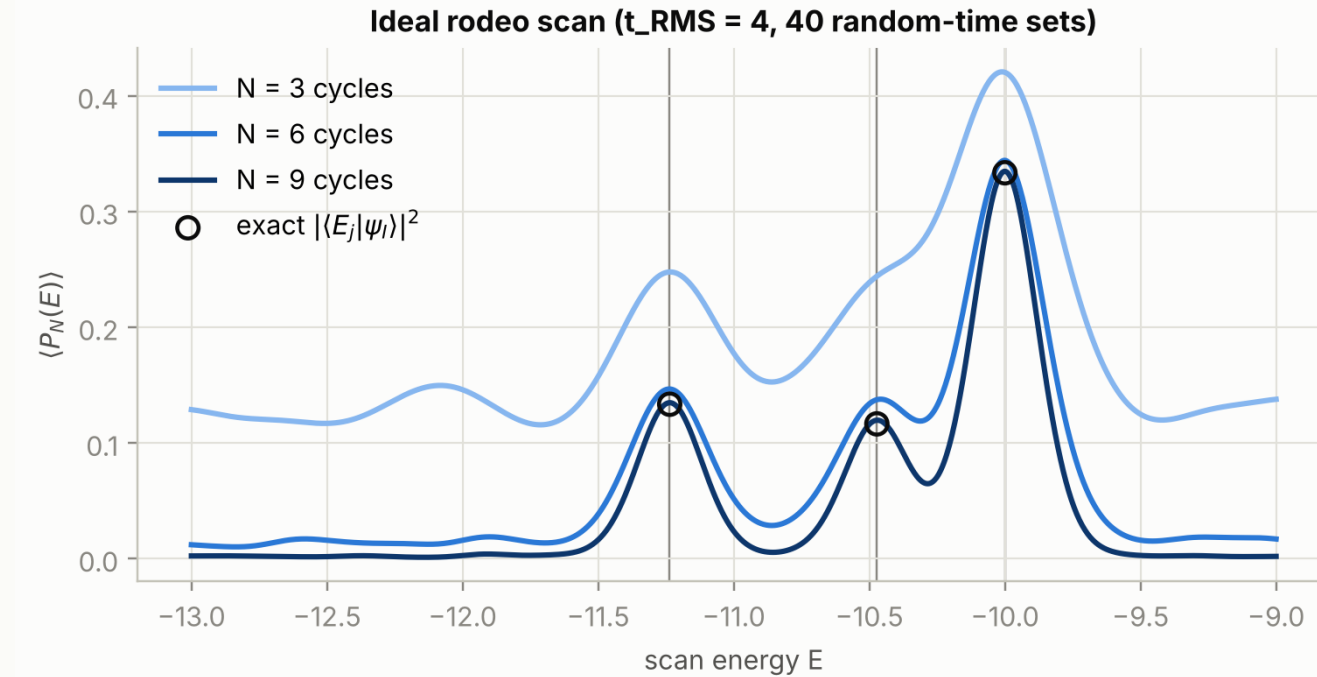
2 • Compensate. The phase gate $P(Et)$ subtracts the energy E being scanned.

3 • Interfere, keep "1". Accept with probability $\cos^2[(E_j - E)t/2]$, reset, repeat N times.

Random Gaussian times stop false revivals; mid-circuit reset means one ancilla for any N .

WHY IT CONVERGES

Exponential suppression, tunable resolution



Suppression

Each cycle keeps a wrong level with probability $\frac{1}{2}$ on average and $\frac{1}{4}$ in a typical run: it dies like 2^{-N} to 4^{-N} .

Resolution

$$\text{FWHM} = \frac{4\sqrt{\ln 2}}{\sqrt{N} t_{\text{RMS}}}$$

Measured widths match within 3 % from $N = 6$ on.

$N = 3$ blurs the levels. By $N = 9$ the peaks sit on the exact energies and their heights reach the overlaps.

DESIGN CHOICE

The trial state decides what you can see

Trial state	E = -11.24	E = -10.47	E = -10.00	in window
Néel $ 01010\rangle$ (paper)	0	0	0	0
$ +\rangle^5$ (5 Hadamards)	0	0	0.031	0.031
$ 11011\rangle$	0.400	0	0	0.400
$ 11111\rangle$	0	0	1.000	1.000
3-term state (used)	0.133	0.116	0.333	0.583

Every gate commutes with H, with magnetisation M and with translations. The filter can only re-weight overlaps that are already there.

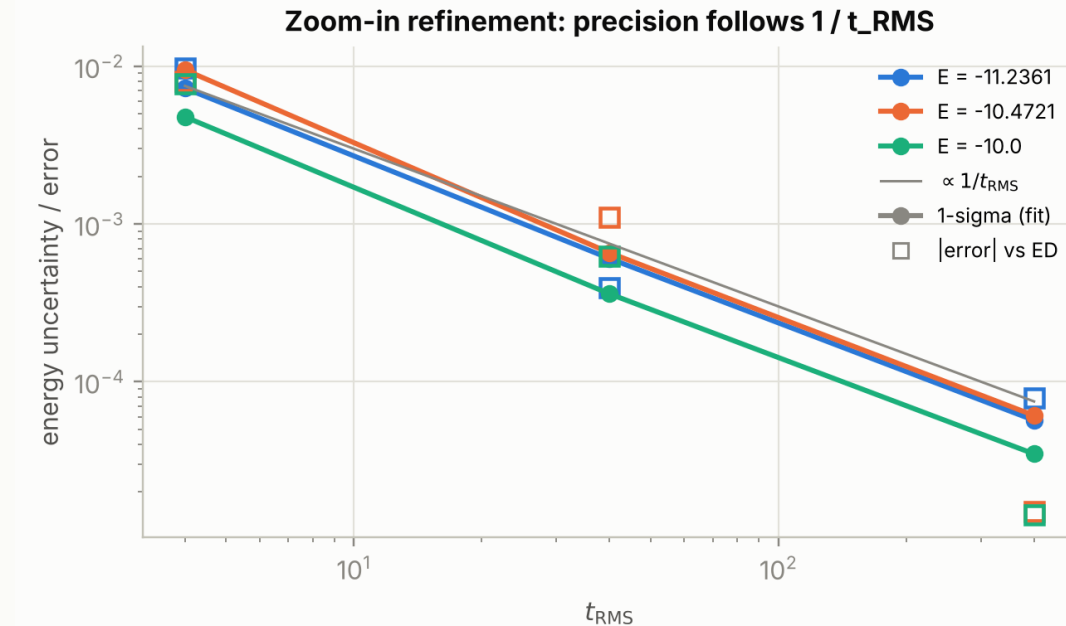
Néel lives in $M = +1$: zero weight in the window.

$|+\rangle^5$ has momentum $k = 0$: it misses both doublets.

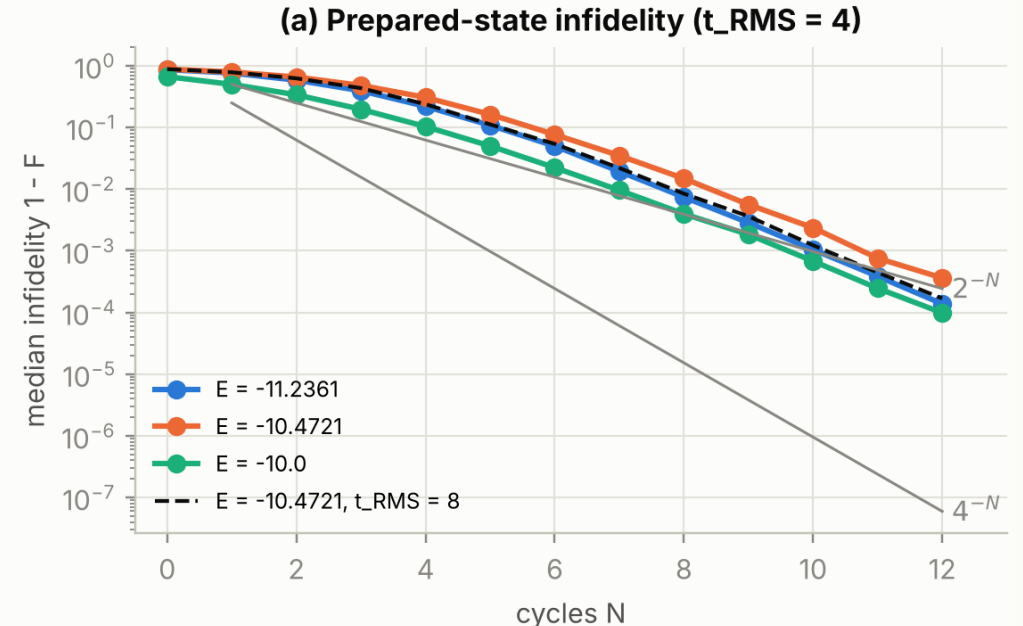
3-term state: 58 % in the window, all three levels, prepared with one CX.

WHAT IT DELIVERS

Heisenberg-limited precision, clean eigenstates



Zoom-in, $t_{\text{RMS}} \times 10$ per stage: σ falls $0.007 \rightarrow 0.0006 \rightarrow 0.00005$, so precision $\sim 1/\text{evolution time}$.



Eigenstate preparation: fidelity above 0.97 after 9 cycles; the price is acceptance falling to the overlap p .

WHAT IT COSTS

The controlled evolution is the whole bill

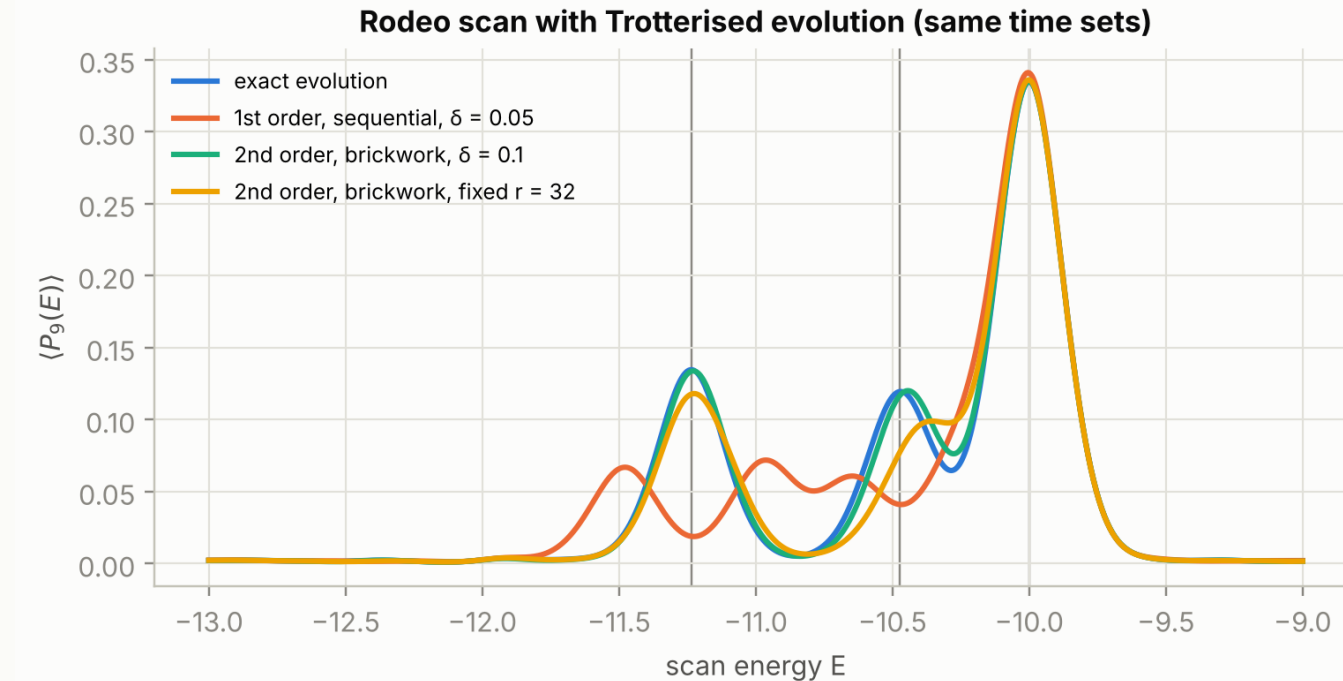
Method (same 40 × 9 random times)	CX / cycle	CX / run	Scan accuracy	Grows with n
A · Exact synthesis (QSD)	1,004	9,036	exact	~ 4^n
B1 · Trotter 1st order, δ = 0.05	2,552	22,972	ghost peaks (0.27)	~ n·t/δ
B2 · Trotter 2nd order, δ = 0.1	1,743	15,683	bias 0.027	~ n·t/δ
C · Constrained depth, D = 9	370	3,330	exact (10^-13)	~ Catalan(n)

$$\vec{\sigma}_j \cdot \vec{\sigma}_k = 2 \text{SWAP}_{jk} - 1$$

Hand-derived controlled exchange gate: 8 CX (Qiskit library 11; our earlier note assumed 12). The field commutes with the exchange terms, so it is applied exactly once per cycle (10 CX).

SURPRISE 1

First-order Trotter splits levels into ghost peaks



Predicted in our earlier note: an $O(\delta)$ shift.

Found: an $O(\delta)$ splitting of each doublet, $\pm 5\delta$ in sequential order.

Why: the leading error term is imaginary-antisymmetric. It vanishes on single levels but separates $+k$ from $-k$.

Fix: second order, error $\sim \delta^2$ (fitted slope 2.00), only 0.027 at $\delta = 0.1$.

Fixing r instead of δ smears the -10.47 level away.

SURPRISE 2

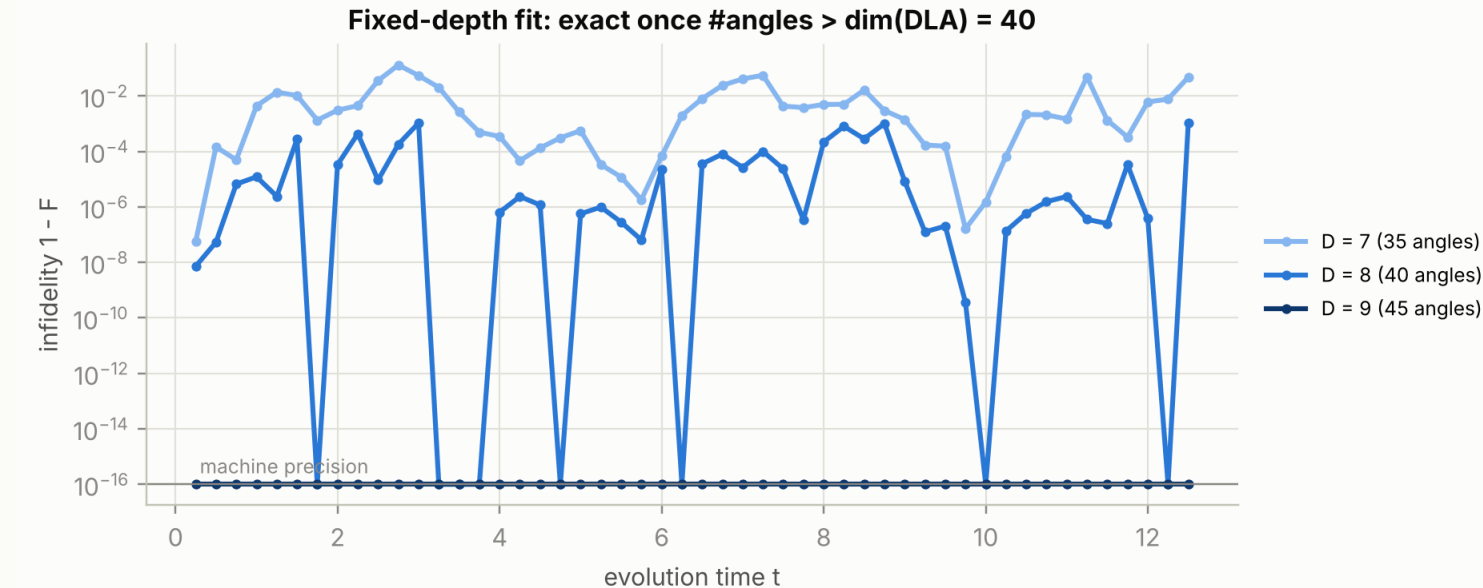
A fixed-depth circuit becomes exact at 45 angles

40

dimension of the dynamical Lie algebra:
 $\mathfrak{su}(4) + \mathfrak{su}(5) + \mathfrak{u}(1)$

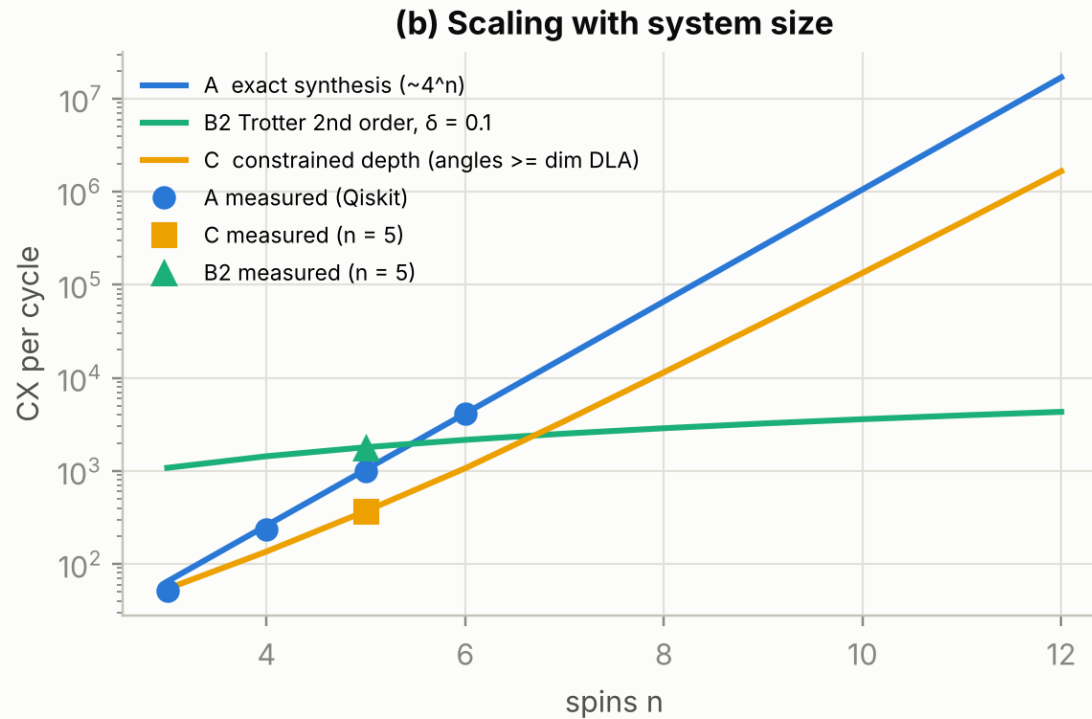
7 layers (35 angles) fail. 8 layers (40) work sometimes. 9 layers (45) are exact at every time.

Exact cycle for 370 CX at any t , versus 1,004 for exact synthesis.



Fitted circuit vs exact evolution; Levenberg–Marquardt with continuation in t .

At five spins, the ranking is a small-size artefact



Trotter is the cheapest compiler from about 7 spins.

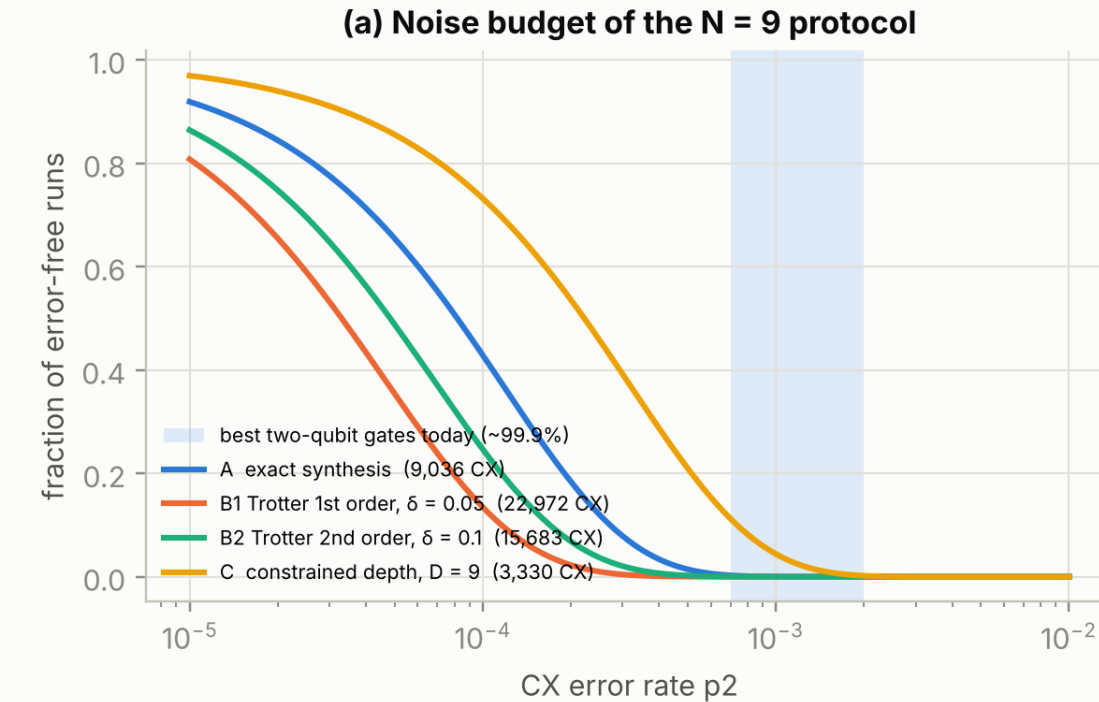
Exact synthesis grows like 4^n : measured $52 \rightarrow 4,140$ CX for $n = 3$ to 6 .

Constrained depth needs as many angles as the Lie algebra has dimensions, $\text{Catalan}(n) - n/2$: also about 4^n .

Both exact compilers need the $2^n \times 2^n$ unitary on a classical computer first.

REALITY CHECK

Noise, and a classical computer that still wins



0.02 %

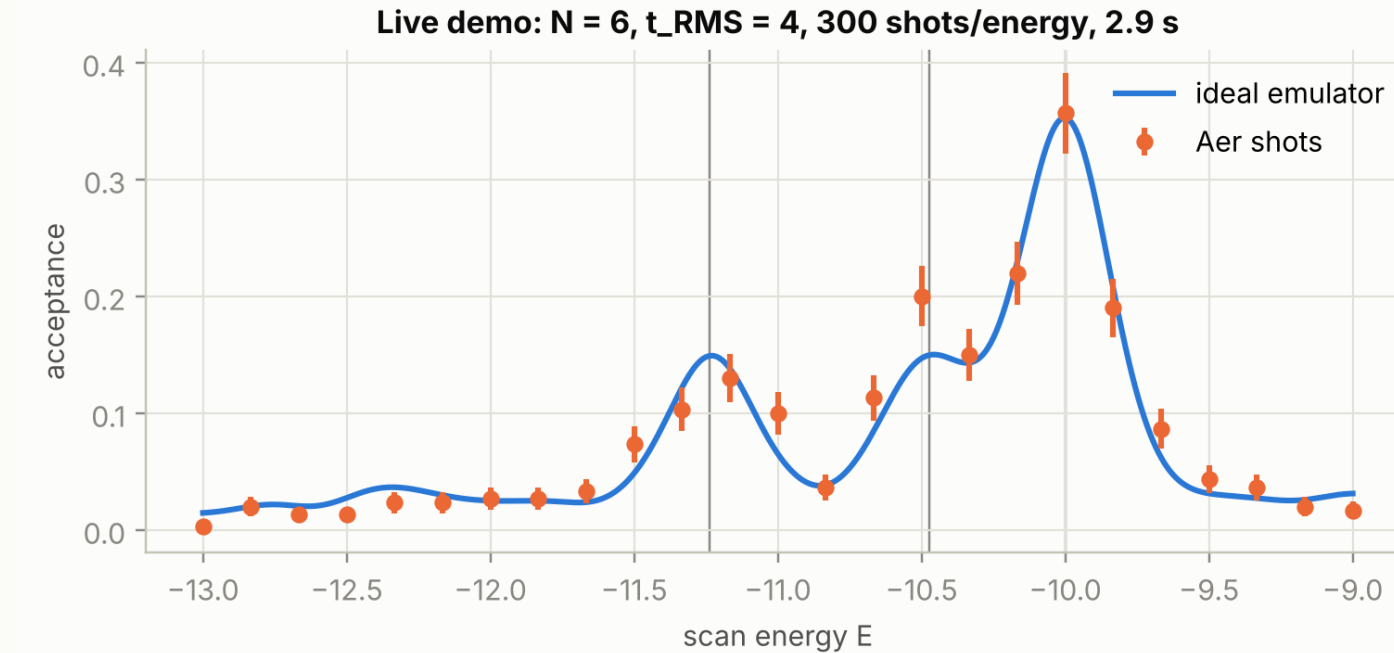
CX error needed for half of the N = 9 runs to be error-free with the best compiler. Today's best gates: about 0.1 %.

A per-cycle error model matches noisy Aer density-matrix runs to within 0.015.

Classical baseline: exact diagonalisation solves $n = 5$ in milliseconds. Lanczos reaches about 18 spins, but interior levels hit the wall first.

LIVE DEMO

Fresh random times, same three levels



```
demo(seed=7)
```

Draws new random times, runs 300 Aer circuits in a few seconds, fits the peaks and prints the estimates with error bars.

Run all in Colab reproduces every number in this talk in about 2 minutes.

TAKEAWAYS

What the rodeo algorithm gives, costs and needs

It works

Exponential filtering with one ancilla.
Levels to 0.00005 and eigenstates above
0.97 fidelity, verified on gate-level
circuits.

It costs

Everything is in the controlled evolution:
an 8-CX exchange gate, ghost peaks at
first order, an exact 370-CX fixed-depth
compiler.

It needs

A trial state with real overlap, patience
for post-selection (4.7 % of runs kept)
and CX errors near 0.02 %.

Next: controlled reversal gates on odd rings · exact local compilation stitched with Trotter · raising the overlap with VQE.

Thank you.